Lieovy závorky, integrabilita a holonomnost

Věta: ϕ_α, ψ_β toky s generátory $a, b \in TM$

$$\phi_\alpha \psi_\beta = \psi_\beta \phi_\alpha \iff [a,b]=0$$

důkaz: $\Xi(\alpha,\beta) = \phi_\alpha \psi_\beta$

$\Rightarrow$

$x \to N_x \quad \Xi(\alpha,\beta)x$ 2 dim podvarieta M

α, β jsou souřadnice na $N_x \to \frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta} \in TN_x$

$a|_{N_x} = \frac{\partial}{\partial\alpha} \qquad b|_{N_x} = \frac{\partial}{\partial\beta}$

$y = \Xi(\alpha,\beta)x \qquad a|_y = \frac{D}{d\varepsilon}\phi_\varepsilon y\big|_{\varepsilon=0} = \frac{D}{d\varepsilon}\underbrace{\phi_\varepsilon\phi_\alpha}_{\phi_{\alpha+\varepsilon}}\psi_\beta x\big|_{\varepsilon=0} = \frac{\partial}{\partial\alpha}\big|_y$

$\left[\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}\right] = 0$

volbou $x \to$ foliace var. M pomocí podvar. N_x

$[a,b] = 0$

$\Leftarrow \quad \phi^*_\alpha = \exp(\alpha\mathcal{L}_a) \qquad \psi^*_b = \exp(\beta\mathcal{L}_b)$

$[a,b]=0 \Rightarrow [\mathcal{L}_a, \mathcal{L}_b] = 0 \Rightarrow [\exp(\alpha\mathcal{L}_a), \exp(\beta\mathcal{L}_b)] = 0$

$\Rightarrow \phi_\alpha\psi_\beta = \psi_\beta\phi_\alpha$

Věta: $a_k \in TU \qquad k = 1\ldots K \qquad U$ okolí $x \in M$

$[a_k, a_l] = 0$ na $U \qquad a_k$ nezávislé

$\Rightarrow$ bodem x prochází podvarieta N_x dim K, ke které jsou a_k tečné

lze zvolit na N_x souřadnice x^k

$a_k = \frac{\partial}{\partial x^k}$

důkaz

${}^k\phi_\alpha$ tok generovaný a_k

2D věta $\Rightarrow \quad {}^k\phi_\alpha \quad {}^l\phi_\beta$ navzájem komutují

$\Xi(\alpha^1, \ldots, \alpha^K) = {}^1\phi_{\alpha^1} \cdots {}^K\phi_{\alpha^K}$

$N_x = \Xi(\alpha^1,\ldots,\alpha^K)x \qquad \alpha^k \in (-\varepsilon, \varepsilon)$

K-dim podvarieta var. M

$x^k \qquad x^k(\Xi(\alpha^1,\ldots,\alpha^K)x) = \alpha^k$ souřadnice na N_x

$\frac{\partial}{\partial x^k}\Big|_y = \frac{D}{d\alpha^k}\Xi(\underbrace{x^1\ldots\alpha^k\ldots x^K}_{\text{fixní souř.}})\big|_{\alpha^k=x^k} = a_k|_y$

Důsledek

$K = \dim M \qquad e_k \qquad [e_k, e_l] = 0 \Rightarrow$

x^k souřad. v $M \qquad e_k = \frac{\partial}{\partial x^k}$

$[e_k, e_l] = 0 \iff e_k$ tvoří holonomní bázi

Věta ϕ_α, ψ_β toky o generátorech $a, b \in TM$

$$[\phi_\tau^*, \psi_\tau^*]\, f = \tau^2 [a,b] \cdot df + O(\tau^3)$$

$$(\phi_\tau^* \psi_\tau^* f - \psi_\tau^* \phi_\tau^* f)|_x$$

$$f(\psi_\tau \phi_\tau x) - f(\phi_\tau \psi_\tau x)$$

důkaz:

$$\varphi(\alpha,\beta) = f(\phi_\alpha \psi_\beta x) \qquad \varphi(0,0) = f(x)$$

$$\psi(\alpha,\beta) = f(\psi_\beta \phi_\alpha x) \qquad \psi(0,0) = f(x)$$

$$\frac{\partial\varphi}{\partial\alpha}(\alpha,\beta) = a \cdot df|_{\phi_\alpha \psi_\beta x} \qquad \frac{\partial\varphi}{\partial\alpha}(0,0) = a \cdot df|_x$$

$$\frac{\partial\varphi}{\partial\beta}(\alpha,\beta) = (\phi_{\alpha*} b) \cdot df|_{\phi_\alpha \psi_\beta x} \qquad \frac{\partial\varphi}{\partial\beta}(0,0) = b \cdot df|_x$$

$$\frac{\partial\psi}{\partial\beta}(\alpha,\beta) = b \cdot df|_{\psi_\beta \phi_\alpha x} \qquad \frac{\partial\psi}{\partial\beta}(0,0) = b \cdot df|_x$$

$$\frac{\partial\psi}{\partial\alpha}(\alpha,\beta) = (\psi_{\beta*} a) \cdot df|_{\psi_\beta \phi_\alpha x} \qquad \frac{\partial\psi}{\partial\alpha}(0,0) = a \cdot df|_x$$

$$\frac{\partial^2\varphi}{\partial\alpha^2}(\alpha,\beta) = a \cdot d(a \cdot df)|_{\phi_\alpha \psi_\beta x} \qquad \frac{\partial^2\varphi}{\partial\alpha^2}(0,0) = a \cdot d(a \cdot df)|_x$$

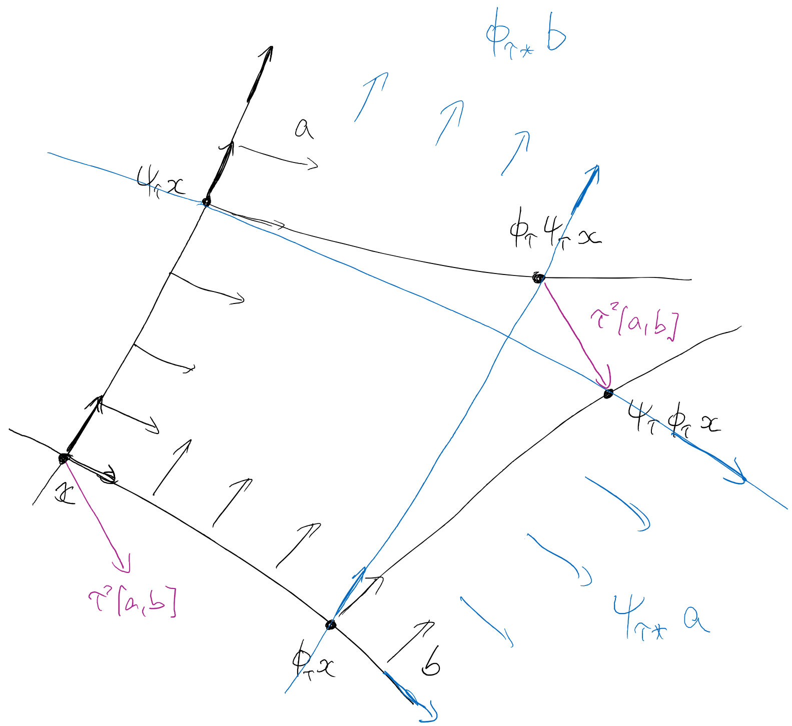
$$\frac{\partial^2\varphi}{\partial\beta^2}(\alpha,\beta) = (\phi_{\alpha*} b) \cdot d(\phi_{\alpha*} b \cdot df)|_{\phi_\alpha \psi_\beta x} \qquad \frac{\partial^2\varphi}{\partial\beta^2}(0,0) = b \cdot d(b \cdot df)|_x$$

$$\frac{\partial^2\varphi}{\partial\alpha\partial\beta}(\alpha,\beta) = (\phi_{\alpha*} b) \cdot d(a \cdot df)|_{\phi_\alpha \psi_\beta x} \qquad \frac{\partial^2\varphi}{\partial\alpha\partial\beta}(0,0) = b \cdot d(a \cdot df)|_x$$

$$\frac{\partial^2\psi}{\partial\beta^2}(\alpha,\beta) = b \cdot d(b \cdot df)|_{\psi_\beta \phi_\alpha x} \qquad \frac{\partial^2\psi}{\partial\beta^2}(0,0) = b \cdot d(b \cdot df)|_x$$

$$\frac{\partial^2\psi}{\partial\alpha^2}(\alpha,\beta) = (\psi_{\beta*} a) \cdot d((\psi_{\beta*} a) \cdot df)|_{\psi_\beta \phi_\alpha x} \qquad \frac{\partial^2\psi}{\partial\alpha^2}(0,0) = a \cdot d(a \cdot df)|_x$$

$$\frac{\partial^2\psi}{\partial\alpha\partial\beta}(\alpha,\beta) = (\psi_{\beta*} a) \cdot d(b \cdot df)|_{\psi_\beta \phi_\alpha x} \qquad \frac{\partial^2\psi}{\partial\alpha\partial\beta}(0,0) = a \cdot d(b \cdot df)|_x$$



$$f(\psi_\tau \phi_\tau x) - f(\phi_\tau \psi_\tau x) = \psi(\tau,\tau) - \varphi(\tau,\tau) =$$

$$= \psi - \varphi + \left(\frac{\partial\psi}{\partial\alpha} + \frac{\partial\psi}{\partial\beta} - \frac{\partial\varphi}{\partial\alpha} - \frac{\partial\varphi}{\partial\beta}\right)\tau +$$

$$+ \left(\frac{1}{2}\frac{\partial^2\psi}{\partial\alpha^2} + \frac{1}{2}\frac{\partial^2\psi}{\partial\beta^2} + \frac{\partial^2\psi}{\partial\alpha\partial\beta} - \frac{1}{2}\frac{\partial^2\varphi}{\partial\alpha^2} - \frac{1}{2}\frac{\partial^2\varphi}{\partial\beta^2} - \frac{\partial^2\varphi}{\partial\alpha\partial\beta}\right)\tau^2 + \dots \Big|_{\substack{\alpha=0\\ \beta=0}}$$

$$= \tau^2 \big(a \cdot d(b \cdot df) - b \cdot d(a \cdot df)\big)|_x + \dots$$

$$= \tau^2 [a,b] \cdot df|_x + \dots$$

Poznámka

$$\phi_{\alpha*}(b|_{\psi_\beta x}) \qquad \phi_{\alpha*} b$$

$$[\phi_{\alpha*} b, a] = 0$$

$$\psi_{\beta*}(a|_{\phi_\alpha x}) \qquad \psi_{\beta*} a$$

$$[\psi_{\beta*} a, b] = 0$$